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Exact vortex solutions of the Navier–Stokes equations with axisymmetric strain and suction or injection

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New solutions of the Navier–Stokes equations are presented for axisymmetric vortex flows subject to strain and to suction or injection. Those expressible in simple separable or similarity form are emphasized. These exhibit the competing roles of diffusion, advection and vortex stretching.

1. Introduction

Vortex dynamics remains a subject of intense research activity, with much modern emphasis on the interaction of vortex structures and the computational modelling of turbulent flows (see e.g. Moffatt 2000; Kerr 2005; Dritschel, Tran & Scott 2007). It is therefore surprising that some rather simple exact solutions have been overlooked. Here we present these solutions which are connected to, or are generalizations of, previous work by Oseen (1911), Bateman (1932), Burgers (1948), Rott (1958), Lundgren (1982), Kambe (1984*a*), Fukumoto (1990) and Moffatt (2000). All of these works are discussed in context.

Throughout this paper, we seek axisymmetric solutions, in cylindrical polar coordinates $(r,\ z)$ and time t, for which the velocity components u and w in the radial r and axial z-directions are prescribed in the form

$$u = Ar + B/r, \quad w = -2Az, \tag{1.1}$$

where, in general, A and B may be constants or functions of time t. Some solutions with $B = 0$ have been given in the above-mentioned papers by Burgers, Rott, Lundgren, Kambe and Moffatt, while solutions with $A = 0$ have been considered by Bateman and Fukumoto. The present emphasis is on solutions for the azimuthal velocity $v(r, t)$, and the corresponding axial vorticity $\omega(r, t)$, which are expressible in relatively simple closed form. Section 2 gives the governing equations and reviews some previous works. Section 3 presents solutions in separable form when both A and B are constants, and § 4 discusses separable solutions where B is constant but A is time-dependent. Section 5 analyses solutions in similarity form, distinguishing between cases with (*a*) $B = 0$, (*b*) $A = 0$ and (*c*) B constant, $A = A(t)$. Section 6 gives a brief physical discussion of these solutions.

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2. Governing equations

Solutions of the incompressible Navier–Stokes equations in cylindrical polar coordinates r, θ, z are presented for flows with axial symmetry. The velocity components are $(u,\ v,\ w)$ in the three coordinate directions, p is the pressure, ρ is the density and ν is the kinematic viscosity. The velocity components and pressure are taken to be functions of $r,\ z$ and time t, but not of the azimuthal angle θ, and body forces are chosen to be zero. Accordingly, the Navier–Stokes equations (see e.g. Drazin & Riley 2006, pp. 8–9) become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{2.1b}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w, \tag{2.1c}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{2.1d}$$

with the axisymmetric Laplacian operator

$$\nabla^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}.$$

When the velocity components u and w have the assumed form (1.1), the continuity equation (2.1d) is satisfied, (2.1a, c) determine the pressure p, and (2.1b) yields

$$\frac{\partial v}{\partial t} + \left(Ar + \frac{B}{r}\right)\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) - 2Az\frac{\partial v}{\partial z} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right). \tag{2.2}$$

Further, since (2.1a, c) together require that v be independent of the axial coordinate z, this reduces to

$$\frac{\partial (rv)}{\partial \tau} = \left(-\gamma r^2 - 2s + r\frac{\partial}{\partial r}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}(rv)\right), \tag{2.3}$$

on using the substitutions $\tau \equiv \nu t$, $A \equiv \nu\gamma$ and $B \equiv 2\nu s$ to eliminate ν. Clearly, the circulation around any circle of radius r is $2\pi r v$. The vorticity ω, wholly directed along the z-axis, is $r^{-1}\partial(rv)/\partial r$, which satisfies

$$\frac{\partial \omega}{\partial \tau} = r^{2s-1}\frac{\partial}{\partial r}\left(r^{1-2s}\frac{\partial \omega}{\partial r}\right) - \gamma\left(r\frac{\partial \omega}{\partial r} + 2\omega\right). \tag{2.4}$$

The corresponding pressure (not required below) is given by

$$\frac{-p}{\rho} = A_t\left(\frac{r^2}{2} - z^2\right) + B_t \log r + A^2\left(\frac{r^2}{2} + 2z^2\right) + B^2\left(\frac{1}{2r^2}\right) - \int_{r_0(t)}^{r} \frac{v^2(r')}{r'}\,\mathrm{d}r',$$

where $A_t,\ B_t$ denote the time derivatives of $A,\ B$ and $r_0(t)$ is an arbitrary non-negative function of time t. Note, too, that (2.3) still holds when the axial velocity component w has the more general form $w = -2Az + W(r)$ compatible with (2.1c). One can work equally well with (2.3) or (2.4); henceforth, we consider solutions of these partial differential equations, focusing on solutions that can be expressed in simple form.

Solutions of (2.4) have been given by Kambe (1984*a*) in cases where $s=0$. Interestingly, these solutions are known when γ is *any* specified function $\gamma(t)$; for, with the transformations (Kambe, 1984*a*, p. 13)

$$\sigma = A(\tau)r, \quad \tau_1 = \int_0^\tau A^2(t')\,\mathrm{d}t', \quad A(\tau) = \exp\left[-\int_0^\tau \gamma(t')\,\mathrm{d}t'\right],$$
$$W(\sigma, \tau_1) = \omega(r, \tau)/A^2(\tau), \tag{2.5}$$

W satisfies the standard diffusion equation

$$W_{\tau_1} = \sigma^{-1}(\sigma W_\sigma)_\sigma, \tag{2.6}$$

where the subscripts denote partial derivatives. Note that $A(\tau)$ defined in (2.5) is not the A introduced previously; to avoid confusion, the latter is henceforth replaced by $\nu\gamma$. Lundgren (1982) and, in other contexts, Kambe (1983, 1984*b*) also used these transformations, and equivalent transformations were first found by Rott (1958). Thereby, from the known solution of (2.6) with arbitrary initial vorticity $\omega(r, 0) = \Omega_0(r)$ (see e.g. Carslaw & Jaeger 1959, p. 259), Kambe finds that

$$\omega(\sigma, \tau_1) = \frac{A^2}{2\tau_1}\exp\left(-\frac{\sigma^2}{4\tau_1}\right)\int_0^\infty \Omega_0(\rho)\exp\left(-\frac{\rho^2}{4\tau_1}\right) I_0\left(\frac{\sigma\rho}{2\tau_1}\right)\rho\,\mathrm{d}\rho, \tag{2.7}$$

where I_0 is the modified Bessel function of the first kind. Kambe notes that when γ is a negative constant, the well-known steady-state solution of Burgers' vortex (see e.g. Drazin & Riley 2006, p. 82) is always approached as $\tau \to \infty$, and also that Oseen's diffusing vortex (Drazin & Riley 2006, p. 169) is a particular solution when $\gamma = 0$ (see also Rott 1958 and Lundgren 1982).

Solutions of (2.3) with $\gamma = 0$ and constant s have been given by Fukumoto (1990). Employing Laplace transforms, he solves the general initial-boundary-value problem for flow outside a cylinder with fixed radius a, with $v(a, t)$ specified for all $t > 0$ and $v(r, 0)$ given for all $r > a$. As might be expected, the solutions are complicated expressions involving double integrals containing Bessel and exponential functions. However, some simplification is found in those special cases where s takes integer values. The details need not be repeated here. But it is worth observing that this problem, of solving for $v(r, t)$ outside a cylinder, is inevitably more complicated than that of the diffusion of initial vorticity, with solution given in (2.7). For, not only does the initial vorticity diffuse with time, but also the boundary at $r = a$ provides a constant source of vorticity that diffuses outwards. Clearly, more general solutions than Kambe's could be constructed, with non-zero γ and with $s = 0$, for flows outside a cylinder of radius a with specified azimuthal velocity $v(a, t)$. Then, the transformed equation (2.6) still applies, and the various solutions for heat conduction given in Carslaw & Jaeger (1959, chapters 7 and 13) are applicable.

3. Separable solutions for γ and s constant

We now consider cases where *both* γ and s are non-zero. These do not appear to have been studied previously. With constant γ and s, let $rv = \mathrm{e}^{-p\tau}F(r)$ in (2.3), so that

$$F'' - \left(\gamma r + \frac{2s+1}{r}\right)F' + pF = 0, \tag{3.1}$$

where primes denote differentiation with respect to r.

When $\gamma = 0$, the resultant equation is closely related to Bessel's equation, and has solutions

$$F(r) = r^{s+1}\left[C_1 J_{s+1}\left(p^{1/2}r\right) + C_2 Y_{s+1}\left(p^{1/2}r\right)\right], \tag{3.2}$$

where C_1 and C_2 are arbitrary constants and J and Y respectively denote Bessel functions of the first and second kind (see Abramowitz & Stegun 1965, p. 362: 9.1.52). (With p replaced by $-p$, modified Bessel functions are instead obtained, as in Fukumoto 1990.) These solutions reduce to simpler forms when $s = N + \frac{1}{2}$, where N is any integer (Abramowitz & Stegun 1965, p. 438), as well as in the trivial case with $p = 0$.

Likewise, with any constant γ and $s = -1/2$, (3.1) reduces to Weber's differential equation

$$y_{xx} - xy_x - ay = 0, \quad x = \gamma^{1/2}r, \quad a = -p/\gamma, \quad \text{where } F(r) = y(x); \tag{3.3}$$

see Kamke (1959, p. 414: 2.44). This has two independent solutions as power series in odd and even powers of x respectively. Though usually infinite, one or the other of these series terminates when a is a negative integer.

In the general case, (3.1) may be re-expressed as the confluent hypergeometric equation (Abramowitz & Stegun 1965, p. 504)

$$xw_{xx} + (b - x)w_x - aw = 0, \quad x = -\gamma r^2/2, \quad b = -s, \quad a = -s + (p/2\gamma), \tag{3.4}$$

where

$$F(r) = \mathrm{e}^{-x}w(x).$$

Again, there are two independent power series solutions for $w(x)$. One series truncates if a is a negative integer, say $-n$ (i.e. $-p/2\gamma = n - s$), and the other series truncates if $(1 + a - b)$ is a negative integer, say $-m$ (i.e. $-p/2\gamma = m + 1$). Clearly, both series truncate if both $-p/2\gamma = m + 1$ and $s = n - m - 1$ for some positive integers m, n.

Without requiring s to be an integer, other closed-form solutions may be found by a different transformation. In (3.1), set

$$F(r) = \exp(\gamma r^2/4) r^{s+1/2} H(r)$$

to obtain

$$H'' = f(r)H, \qquad f(r) \equiv \frac{\left(s + \frac{1}{2}\right)\left(s + \frac{3}{2}\right)}{r^2} + (\gamma s - p) + \frac{\gamma^2 r^2}{4}.$$

Setting $U(r) \equiv H'/H$ then gives a differential equation of Riccati type,

$$U' + U^2 = f(r).$$

This has particular solutions

$$U = \frac{\bar{a}}{r} + \bar{b}r$$

in the following four cases:

$$(\bar{a}, \bar{b}) = \left(s + \tfrac{3}{2}, -\tfrac{1}{2}\gamma\right),\ \left(s + \tfrac{3}{2}, \tfrac{1}{2}\gamma\right),\ \left(-s - \tfrac{1}{2}, -\tfrac{1}{2}\gamma\right),\ \left(-s - \tfrac{1}{2}, \tfrac{1}{2}\gamma\right),$$

provided $\bar{b}(1 + 2\bar{a}) = (\gamma s - p)$. These respectively require that $p = 2\gamma(1 + s)$, -2γ, 0 and $2\gamma s$, and the corresponding solutions for rv are

$$K\exp(-2\gamma(1 + s)\tau)r^{2(s+1)}, \quad K\exp\left(2\gamma\left(\tau + \tfrac{1}{4}r^2\right)\right)r^{2(s+1)}, \quad K,$$
$$K\exp\left(2\gamma\left(-s\tau + \tfrac{1}{4}r^2\right)\right), \tag{3.5}$$

where K is any constant. Note that, provided γ is negative, the second and fourth of these solutions decay to zero as r approaches infinity; that the second grows or decays as time τ increases according to whether γ is positive or negative; and that the fourth grows or decays as time τ increases according to whether γs is negative or positive. When $s=0$, the difference of the third and fourth solutions together yield the Burgers vortex (see (5.1) below). (There will of course be other particular solutions of (3.1), but none having $U(r)$ with the simple form above.)

Finally in this section, we observe that when solutions $F(r;\,p)$ are known for all p, a complete formal solution of the general initial-value problem for rv may be constructed by Laplace transforms. Inevitably, the result yields complicated expressions: those with $\gamma\ =0$ involve Bessel functions and are fully discussed by Fukumoto (1990). Also, as mentioned above, the initial-value problem for vorticity when $s=0$ was solved by Kambe (1984*a*). The general solution with non-zero s and γ contains integrals involving hypergeometric functions. In just a few special cases, reduction to simple expressions is possible. For instance, if $\gamma\ =0$, the C_1 solution of (3.2) gives

$$rv=\int_0^\infty K(p)\mathrm{e}^{-p\tau}F(r;p)\,\mathrm{d}p,\quad F(r;p)=r^{s+1}J_{s+1}\big(p^{1/2}r\big), \tag{3.6}$$

i.e.

$$rv=2r^{s-1}\int_0^\infty K(\zeta^2/r^2)\exp(-a^2\zeta^2)\zeta\,J_{s+1}(\zeta)\,\mathrm{d}\zeta,$$

with $\zeta\equiv p^{1/2}r$ and $a^2\equiv\tau/r^2$. Choosing the particular case $K(\zeta^2/r^2)=(2\zeta/r)^{s+1}$ gives a known integral (Abramowitz & Stegun 1965, p. 486: 11.4.29), namely

$$rv=r^{2s+2}\tau^{-s-2}\exp(-r^2/4\tau), \tag{3.7}$$

a solution that is derived more simply in the following section. Clearly, many more such solutions may be constructed for other choices of $K(p)$.

4. Separable solutions for any $\gamma(\tau)$ and constant s

For any $\gamma(\tau)$ and $s(\tau)$, (2.4) and (2.5) yield the vorticity equation in transformed variables as

$$W_{\tau_1}=\sigma^{2s-1}(\sigma^{1-2s}W_\sigma)_\sigma, \tag{4.1}$$

where the subscripts denote partial differentiation; this resembles (2.4) with $\gamma\ =0$. The corresponding equation for rv is

$$\frac{\partial}{\partial\tau_1}(rv)=\left(\frac{\partial}{\partial\sigma}-\frac{2s+1}{\sigma}\right)\frac{\partial}{\partial\sigma}(rv), \tag{4.2}$$

which is just (4.1) with W replaced by rv and s replaced by $s+1$. Though these equations apply when s is a function of time, in this section we henceforth restrict attention to cases where s is constant.

A further change of variables to

$$\rho\equiv\sigma^{2(1-s)},\quad \tau_2\equiv4(1-s)^2\tau_1,\quad W(\tau_1,\sigma)\equiv\Xi(\tau_2,\rho)$$

transforms (4.1) into

$$\Xi_{\tau_2}=(\rho^\mu\,\Xi_\rho)_\rho,\quad \mu\equiv\frac{1-2s}{1-s},\quad (s\neq1). \tag{4.3}$$

This is identical to the equation for heat conduction in a bar with conductivity that varies as the μth power of distance.

Solutions in the form $\Xi = e^{p\tau_2} F(\rho;\mu)$ are given by Carslaw & Jaeger (1959, p. 413), where F is

$$\rho^{(1/2)(1-\mu)} Z_\nu \left(\frac{2p^{(1/2)}\rho^{1-(1/2)\mu}}{2-\mu} \right), \quad \nu \equiv \left(\frac{1-\mu}{2-\mu} \right), \ (\mu \neq 2), \tag{4.4}$$

and Z_ν denotes the modified Bessel functions I_ν or K_ν.

When $s=1$, (4.1) becomes

$$W_{\tau_1} = \sigma(\sigma^{-1} W_\sigma)_\sigma, \tag{4.5}$$

and the substitution $\sigma \equiv \exp x$, where $-\infty < x < \infty$ yields

$$W_{\tau_1} = (e^{-2x} W_x)_x,$$

which is the equation for heat conduction in a bar with conductivity varying as $\exp(-2x)$. But (4.5) then directly yields the separable solutions

$$W = e^{p\tau_1} \sigma Z_1 \left(p^{1/2}\sigma \right), \tag{4.6}$$

where Z_1 denotes the modified Bessel functions I_1 or K_1 (see e.g. Kamke 1959, p. 440: result 2.162 (9)).

Recall that W is related to the vorticity by $\omega = A^2(\tau)W$, where $A(\tau)$ and other scaled quantities are defined in (2.5). In order to find the corresponding azimuthal velocity v, it is necessary to perform a further integration, using

$$r^{-1}\partial(rv)/\partial r = \omega(r,t)$$

in the original variables. In transformed variables, this gives

$$rv = \int_{\sigma_0}^{\sigma} \sigma' W(\tau, \sigma')\, d\sigma' \tag{4.7}$$

with arbitrary lower limit of integration. Or, more simply, to find the corresponding rv to within an additive and multiplicative constant, one may replace s by $s+1$ in expressions for $W(\tau_1, \ \sigma)$, as explained at (4.2). Clearly, Bessel functions will usually result. Although the transformed equations (4.1) and (4.2) continue to hold when s is a function of time, no simple solutions have then been found.

5. Similarity solutions with constant *s*

The separable solutions discussed above comprise the building blocks of integral-transform representations of the solutions to general axisymmetric initial-boundary-value problems. Unfortunately, these general solutions seldom reduce to compact expressions. However, there exist various other special solutions in compact form, which we now discuss.

5.1. *Cases with* $s=0$

Perhaps the best known such solution is the steady Burgers vortex (Burgers 1948), for which $s=0$ and γ constant, say $-k/\nu$. Then,

$$rv = \Gamma(1 - \exp(-kr^2/2\nu)), \quad \omega = \frac{\Gamma}{\nu} \exp(-kr^2/2\nu), \tag{5.1}$$

where the circulation at infinity is $2\pi\Gamma$ (see e.g. Drazin & Riley 2006, pp. 82–83). (A more elaborate steady 'two-cell' vortex solution due to Sullivan (1959) – also described by Drazin & Riley (2006, p. 83) – falls outwith the scope of the present paper since the velocity components u, w differ in form from those assumed here.)

A time-dependent solution that approaches the Burgers vortex at large times t was obtained by Kambe (1984*a*, equation (10)) as

$$\omega = \frac{\Gamma/\nu}{1-\exp(-2kt)} \exp\left(\frac{-kr^2/2\nu}{1-\exp(-2kt)}\right). \tag{5.2}$$

This is closely connected to Oseen's (1911) solution for a diffusing line vortex with $\gamma = 0$, namely

$$\omega = \frac{\text{constant}}{4\nu t} \exp\left(\frac{-r^2}{4\nu t}\right). \tag{5.3}$$

This satisfies (2.6) when $(r, \nu t)$ are replaced by (σ, τ_1), and the transformations (2.5) with constant $\gamma = -k/\nu$ immediately yield (5.2). This solution was also stated by Gibbon, Fokas & Doering (1999, equation (45)). Much earlier, Rott (1958, equation (29)) found related solutions for the circulation,

$$rv = \Gamma_\infty \left\{1 - \exp\left(\frac{-kr^2/2\nu}{1+\beta\exp(-2kt)}\right)\right\}, \tag{5.4}$$

where β is any constant. If β is negative, this merely reinitializes the time origin of solution (5.2); but positive β values give another class. Rott also observed that differentiating (5.4) with respect to time t yields further solutions.

Many other such solutions with $s = 0$ and $\gamma = \gamma(t)$ may be constructed by using the transformations (2.5). Solutions with γ proportional to $(t_0 - t)^{-1}$ are given by Moffatt (2000) and also discussed in Drazin & Riley (2004, pp. 169–171); these are mentioned further below. Also, as pointed out by Drazin & Riley (p. 170), there are solutions of the form

$$rv = C\left(1 - \mathrm{e}^{-\zeta^2}\right) \tag{5.5}$$

for *any* prescribed $\gamma(t)$, for a suitably defined similarity variable ζ (which is just $r/(4\nu t)^{1/2}$ in the classical Oseen case).

5.2. *Cases with $\gamma = 0$, $s \neq 0$*

Perhaps the earliest similarity solution with non-zero constant s, but zero γ, is that of Bateman (1932, p. 349), which is little known. Even earlier, Bateman published his solution in an *NACA Report* (Bateman 1923), then with restricted circulation. Posed as an example in his text on partial differential equations, Bateman asks his readers to show that (in our notation)

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial}{\partial r} - \frac{2s}{r}\right)\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) \tag{5.6}$$

and that this has a solution of the type

$$v = r^{2s+1} t^{-s-2} \exp(-r^2/4\nu t). \tag{5.7}$$

This result is precisely solution (3.7) above. (In his next example, Bateman asks for solutions as infinite series, involving ascending powers of $(r^2/\nu t)$, which he connects with the confluent hypergeometric function in a particular case.)

Note, also, that (5.6) has the steady solutions

$$v = \frac{C_1}{r} + C_2 r^{2s+1}, \tag{5.8}$$

where C_1 and C_2 are arbitrary constants.

Further similarity solutions follow from the simple observation that any integral or derivative of v with respect to t, holding r fixed, is also a solution of (5.6), except at any singular points; and, of course, any linear combination of such solutions is also a solution.

A more systematic approach is to seek solutions in the form

$$rv = t^{\alpha} F(\eta), \quad \eta \equiv r^2/4\nu t, \tag{5.9}$$

where, from (5.6) or (2.3) with $\gamma = 0$, F must satisfy

$$\eta(F'' + F') = sF' + \alpha F. \tag{5.10}$$

Bateman's solution (5.7) corresponds to the choice $\alpha = -1$ and $F = \eta^{s+1}\exp(-\eta)$. Perhaps the simplest is $\alpha = s$, which leads to

$$F(\eta) = \exp(-\eta)\left[C_1 + C_2 \int_{\eta_0}^{\eta} u^s \mathrm{e}^u \,\mathrm{d}u\right], \tag{5.11}$$

where $rv = t^s F(\eta)$, C_1 and C_2 are arbitrary constants and η_0 is chosen so that the integral converges. Note that the C_1 solution decays exponentially as η approaches infinity, but the C_2 solution does not.

As just mentioned, differentiating or integrating such known solutions with respect to t, while holding r fixed, gives further solutions of (5.6) in closed form. For instance, repeatedly differentiating

$$rv = t^s \exp(-a/t) \equiv Q_0(t; a, s), \quad a \equiv r^2/4\nu, \tag{5.12}$$

corresponding to the C_1 solution in (5.11), gives the further solutions $rv = Q_n(t; a)$, where

$$\begin{aligned}
Q_1 &= t^{s-2}(a + st)\exp(-a/t),\\
Q_2 &= t^{s-4}[a^2 + 2a(s-1)t + s(s-1)t^2]\exp(-a/t),\\
Q_3 &= t^{s-6}[a^3 + 3a^2(s-2)t + 3a(s-1)(s-2)t^2 + s(s-1)(s-2)t^3]\exp(-a/t),\\
&\vdots\\
Q_n &= \exp(-a/t)\sum_0^n t^{s-n-r}\,{}_nC_r\frac{\Gamma(s+1-r)}{\Gamma(s+1-n)}a^r.
\end{aligned} \tag{5.13}$$

In the last expression, $\Gamma(x)$ denotes the gamma function and ${}_nC_r$ the binomial coefficient $n!/r!(n-r)!$. Note also the useful reduction formula

$$\frac{\mathrm{d}Q_0\,(t; a, s)}{\mathrm{d}t} = sQ_0\,(t; a, s-1) + aQ_0\,(t; a, s-2) \tag{5.14}$$

that aids repeated differentiation.

These solutions are related to Bateman's solution (5.7); for, provided s is a positive integer, (5.7) is equal to the $(s+1)$th t-derivative of the above Q_0, times the constant $(4\nu)^{s+1}$. (To see why, note that all but the last term vanishes in the sum for the appropriate Q_n.) But Bateman's solution cannot be recovered by differentiation or

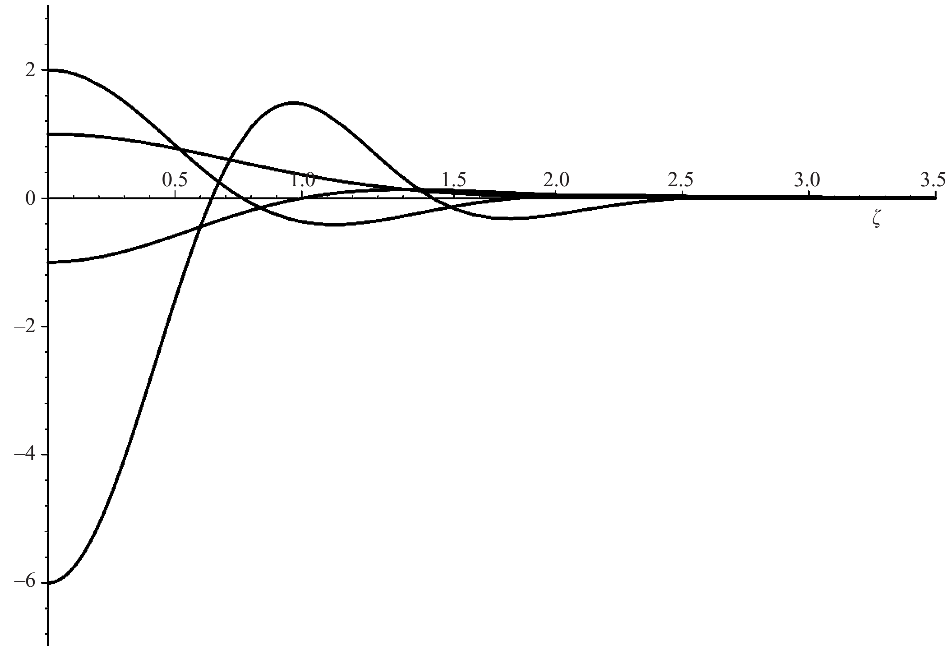

FIGURE 1. The functions $Q_n t^{n-s}$ of (5.13) versus $\zeta = r/(4\nu t)^{1/2}$ for $n = 0, 1, 2, 3$ with $s = -1$, showing some reversals of azimuthal velocity. On $\zeta = 0$, these take the respective values 1, −1, 2 and −6.

integration of (5.12) when s is a non-integer. In such cases, a new set of solutions may be found from Bateman's solution (5.7) by differentiating or integrating with respect to t in a similar manner.

In this connection, we note that the similarity solution found by choosing $\alpha = 0$ in (5.9) is just that obtained on integrating Bateman's solution with respect to time t, namely

$$rv = C \int_K^\eta u^s \mathrm{e}^{-u}\, \mathrm{d}u, \tag{5.15}$$

where C and K are arbitrary constants and η is as defined in (5.7).

It is worth observing that the solutions of (5.13) typically exhibit flow reversal where the sign of v changes. For instance, in Q_1, this happens at $r^2/4\nu t = -s$ if s is negative (and supposing t is positive). Similarly, Q_2 exhibits no reversals if $s > 1$, one reversal if $0 < s < 1$ and two if s is negative; and Q_3 has no reversals if $s > 2$, one if $1 < s < 2$, two if $0 < s < 1$ and three if $s < 0$. Clearly, there exist similarity solutions with many reversals. The functions $Q_n t^{n-s}$ depend only on a/t; these are shown in figure 1 for $n = 0, 1, 2, 3$ with $s = -1$, where the abscissa $(a/t)^{1/2} = r/(4\nu t)^{1/2}$ is proportional to the radius r. The flow reversals are evident.

Though attention has largely focused on solutions that decay to zero as r approaches infinity, those that do not are of interest in contexts where the fluid is confined within finite regions. Then, the C_2 solution of (5.11) becomes relevant. Choosing

$$\frac{rv}{t^s} = \mathrm{e}^{-\eta} \int_0^\eta u^s \mathrm{e}^{u}\, \mathrm{d}u \equiv \mathrm{e}^{-\eta} I(\eta; s) \tag{5.16}$$

(which is appropriate for all $s > -1$), an integration by parts gives

$$I(\eta;s) = \eta^s e^\eta - sI(\eta;s-1). \tag{5.17}$$

Repeated integration by parts, say N times, generates polynomial terms in η, together with an integral $I(\eta;\ s-N)$, all multiplied by $\exp(-\eta)$; and if s is the positive integer N, the resulting integral $I(\eta;\ 0)$ equals $\exp(\eta) - 1$. The -1 term may be discarded, as it corresponds to the decaying C_1 solution of (5.11), and there remains a set of algebraic solutions for $s = N(=1, 2, 3, 4, \dots)$:

$$\frac{rv}{t^N} = \eta^N - N\eta^{N-1} + N(N-1)\eta^{N-2} + \cdots + (-1)^r \frac{N!}{(N-r)!}\eta^{N-r} + \cdots + (-1)^N N!. \tag{5.18}$$

This expression is just $(-1)^N N!$ times the $(N+1)$th partial sum of $e^{-\eta}$, and is closely related to exponential integrals. However, if s is a non-integer, an integral term in $I(\eta;\ s-N)$ will always remain. The expression (5.16) has no positive roots η, and so does not represent a flow where v changes sign. However, the set of solutions generated by the various t-derivatives of this rv, with r held fixed, may well do so, as did the solutions (5.13) above. Similar solutions may be obtained for $s \leqslant -1$ by choosing a positive lower limit of integration in (5.16).

When $\alpha = -1$, a second independent solution, in addition to Bateman's (5.7), is readily found. Then, (5.10) has the first integral

$$\eta F' + (\eta - s - 1)F = C, \tag{5.19}$$

where C is constant. Taking $C = 0$ gives $F(\eta) = K\exp(-\eta)\eta^{1+s}$ for any constant K, which is just Bateman's solution (5.7). But the general solution of (5.19) gives

$$rv = Ct^{-1}e^{-\eta}\eta^{s+1}\int_k^\eta u^{-s-2}e^u\,du, \quad \eta \equiv r^2/4\nu t \tag{5.20}$$

with arbitrary lower limit of integration k. Again, repeated integration by parts yields polynomials in η, which terminate if s is an integer.

We have focused on those similarity solutions corresponding to $\alpha = -1, \alpha = s$ and, briefly, $\alpha = 0$. Despite the apparent simplicity of (5.10), few other choices of α lead to simple solutions. The next simplest is $\alpha = s/2$, for which

$$F(\eta) = \eta^\nu e^{-\eta/2} Z_\nu(\eta), \quad \nu \equiv \frac{s+1}{2}, \tag{5.21}$$

where $Z_\nu(\eta)$ denotes the modified Bessel functions $I_\nu(\eta)$ and $K_\nu(\eta)$ (cf. Kamke 1959, p. 441: 2.162, no. 18).

5.3. *Cases with* $\gamma = \gamma(t), s \neq 0$

Similarity solutions of form (5.9) exist only if $\gamma(t)$ varies as t^{-1}. Setting $\gamma(t) = Gt^{-1}$ gives

$$\eta(F'' + (1+2G)F') = sF' + \alpha F \tag{5.22}$$

in place of (5.10), and the choice $\alpha = s\,(1+2G)$ yields the first integral

$$F' + (1+2G)F = C_1\eta^s,$$

where C_1 is an arbitrary constant. Hence, the general solution in this form is

$$\frac{rv}{t^{s(1+2G)}} = C_2 e^{-\eta_1}\int_k^{\eta_1} u^s e^u\,du, \quad \eta_1 \equiv (1+2G)\eta, \tag{5.23}$$

where C_2 and k are arbitrary constants.

This reduces to the solution (5.11) with $\gamma = 0$. Also, when $s = 0$, it becomes

$$rv = C_3 + C_2 \exp\left(\frac{-(1+2G)r^2}{4\nu t}\right), \tag{5.24}$$

which is essentially the solution found by Moffatt (2000, equation (2.20)). (His solution has $C_3 = -C_2, 2G = -c$ and t replaced by $t - t^*$. When $c > 1$, his corresponding solution (2.14) for the vorticity remains bounded for all $t < t^*$ and becomes singular on the axis $r = 0$ at t^*.) Clearly, if $G = 0$, (5.24) yields Oseen's diffusing vortex solution (5.3).

In fact, a class of solutions with $s = 0$ that is more general than (5.24) comes from applying the transformations (2.5) with $\gamma(t) = Gt^{-1}$; but first it is necessary to alter the two lower limits of integration from zero to finite quantities. Sparing details, the end result is

$$rv = C_3 + C_2 \exp\left(\frac{-(1+2G)r^2}{4\nu}\frac{t^{2G}}{t^{2G+1} - t_1^{2G+1}}\right), \tag{5.25}$$

where t_1 is an arbitrary constant deriving from the lower limit of integration for τ_1 in (2.5). (Taking $t_1 = 0$ gives (5.24) again.) When $1+2G<0$, the corresponding vorticity remains bounded when $t_1<t<0$ and decays to zero at large r; but it is singular on the $r=0$ axis at *both* $t=t_1$ and $t=0$. One may think of the singularity at t_1 as imposed by initial data, and that at $t=0$ as corresponding to Moffatt's singularity when $\gamma(t)$ becomes infinite. Note that unless $C_2+C_3=0$, the azimuthal velocity v is singular at $r=0$ for all t.

When s is non-zero, many more solutions with non-zero $\gamma(t)$ may be found by applying the transformations (2.5) to those solutions with $\gamma=0$ which are given in the preceding section. For instance, those deriving from (5.11) when γ is a constant are

$$rv = \left[\frac{[1-\exp(-2\gamma\nu t)]}{2\gamma}\right]^s \mathrm{e}^{-\xi}\left[C_1 + C_2\int_{\xi_0}^{\xi} u^s \mathrm{e}^u\,\mathrm{d}u\right], \quad \xi \equiv \frac{-\gamma r^2}{2[1-\ \exp(2\gamma\nu t)]}, \tag{5.26}$$

where $C_1,\ C_2$ and ξ_o are arbitrary constants, ξ_o being chosen so that the integral converges. (When $s = 0$, this gives solution (5.4).)

However, rather than applying the transformations (2.5) which are expressed in terms of vorticity, one may use an alternative approach based directly on *rv* that is broadly equivalent in the present context. This is close to the method of Rott (1958) as described in Drazin & Riley (2006, pp. 169–170) for cases with $s = 0$. (The author is most grateful to a referee for drawing this to his attention and for outlining the analysis of the following paragraph.)

Similarity solutions of (2.3) are sought with the more general form $rv = \Psi(\zeta)$, where $\zeta = r/\delta(\tau)$. Equation (2.3) then yields

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\zeta^2} + \left\{(\delta\,\dot{\delta} - \gamma\delta^2)\zeta - \frac{2s+1}{\zeta}\right\}\frac{\mathrm{d}\Psi}{\mathrm{d}\zeta} = 0, \tag{5.27}$$

where the overdot denotes differentiation with respect to τ. On choosing $\delta\,\dot{\delta} - \gamma\delta^2 = 2s+1$, which determines δ, Ψ satisfies

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\zeta^2} + (2s+1)\left(\zeta - \frac{1}{\zeta}\right)\frac{\mathrm{d}\Psi}{\mathrm{d}\zeta} = 0,$$

from which it follows that

$$\frac{d\Psi}{d\zeta} = C\zeta^{2s+1} \exp\left[-\left(s+\frac{1}{2}\right)\zeta^2\right] \qquad (\zeta = r/\delta(\tau),\ C \text{ constant}) \tag{5.28}$$

and

$$\delta^2(\tau) = (4s+2)\int_{\tau_0}^{\tau} \exp\left(2\int_{\tau_1}^{\tau_2} \gamma(t')\,dt'\right) d\tau_2, \tag{5.29}$$

where τ_0 and τ_1 are arbitrary constants. When $s = 0$ the corresponding rv given by (5.28)–(5.29) includes the particular solution (5.5) already mentioned. Also, if $\gamma = 0$ and s is non-zero, the corresponding solution for rv simplifies to (5.15). As this is just one member of a family of similarity solutions (corresponding to the choice $\alpha = 0$ in (5.9)), it is clear that there are further similarity solutions to be found for non-zero $\gamma(t)$.

This may be done by setting $rv = \mathscr{F}(\tau)\Psi(\zeta)$ with

$$\mathscr{F}(\tau) = \exp\int_{\tau_0}^{\tau} K\delta^{-2}(\tau')\,d\tau' \qquad (K\text{constant})$$

and $\delta(\tau)$ as above. Then, the equation for $\Psi(\zeta)$ becomes

$$\frac{d^2\Psi}{d\zeta^2} + (2s+1)\left(\zeta - \frac{1}{\zeta}\right)\frac{d\Psi}{d\zeta} = K\Psi$$

in place of that above. This may be transformed into a form like (5.10), yielding new classes of solutions corresponding to particular choices of K, much as above. Also, new solutions may be found as time derivatives or time integrals of those already known, as already discussed. The details are suppressed.

6. Discussion

It must be admitted that the direct practicality of many of the above solutions is rather limited. For, the imposed straining flow represented by γ becomes ever larger at large distances z or r, and terms in s require either an infinite line source or sink along the z-axis, or cylindrical boundaries at specified radius r where there is appropriate suction or injection. But such idealization permits mathematical simplicity that aids physical understanding and well represents the local behaviour of more complex flows. Unfortunately, the general solutions of initial-boundary-value problems are too complicated to give much physical understanding except in asymptotic limits, such as when t becomes very large (see e.g. Fukumoto 1990); but even these have value in revealing whether or not the circulation $2\pi rv$ decays to zero as r approaches infinity. Our simpler solutions, besides their intrinsic interest, may well be of use in checking computer codes for direct numerical simulations.

The physical processes acting in our simple geometrical configuration are threefold. There is viscous diffusion; radial advection associated with the inflow or outflow velocity component u, deriving from either or both of the imposed straining field and suction or injection of fluid; and vortex stretching or contraction due to the imposed straining velocity component w. The interplay of these three processes can yield various outcomes.

With neither externally imposed strain nor suction or injection ($s = \gamma = 0$), a line vortex with an initial singularity along the z-axis at $t = 0$ diffuses as in Oseen's solution (5.3) when $t > 0$. But, when γ is a negative constant, the radial inflow u and vortex stretching by w counteracts viscous diffusion so that a final steady state is reached: the well-known Burgers vortex (see (5.1), (5.2), (5.4) above). On the other hand, if γ is time-dependent, there is no final steady state. For instance, when $\gamma(t) = \Gamma t^{-1}$, Moffatt's (2000) solution (cf. (5.24) above) starts with a bounded vortex with Gaussian shape in r at some negative t, and this becomes singular at $t = 0$ (equivalently, Moffatt's t^* with $\gamma(t) = \Gamma(t - t^*)^{-1}$). This was generalized in (5.25) to give a solution for which the vorticity has a singularity on the z-axis at $t = t_1 < 0$, corresponding to an inviscid line vortex as the initial condition at that time, and which is subsequently finite up to $t = 0$. At this time, the vortex again becomes singular, since the ever-increasing inflow and vortex-stretching due to $\gamma(t) = \Gamma t^{-1}$ reverses the effect of all the viscous diffusion that has taken place.

When s is a non-zero constant and $\gamma = 0$, the C_1 part of solution (5.9), (5.11) yields Oseen's diffusing vortex when s tends to zero. But when $s > 1$ the corresponding vorticity, $\omega = C_1(t^{s-1}/2\nu)\exp(-r^2/4\nu t)$, has no singularity, as it is instead zero everywhere at $t = 0$. The subsequent vorticity in the flow results from injection at the 'inner boundary' $r = 0$, combined with viscous diffusion. More precisely, if, instead of suction or injection on the axis $r = 0$, we envisage equivalent conditions to be applied at a circular boundary with very small non-zero radius r_0, the imposed radial and azimuthal velocities there would be $u = 2\nu s/r_0$, $v = C_1 t^s/r_0$, and the corresponding vorticity ω would be $C_1(t^{s-1}/2\nu)$, for all but very small times $t > 0$. Note that these 'boundary values' of v and ω respectively increase or decrease with time t according to whether s is greater or less than 0 and 1; these changes in azimuthal velocity and vorticity are imposed not by the blowing or suction as such, but by the requirement of self-similarity. Obviously, these boundary values have a large influence on the developing flow.

Their role is made plain in figure 2, which shows the simple solution (5.12) for various values of s and time t. To better display the comparison, the solutions are rescaled so that they are identical at the initial time $\tau = 4\nu t = 0.1$, and they are shown also at the later times $\tau = 0.2$, 0.5 and 1.0 for $s = 0$, 1 and -1. The positive s values correspond to 'blowing' and the negative s values to 'suction' at the axis $r = 0$. It is clear that, on top of the contribution of viscous diffusion, solutions with blowing grow in strength by injection of vorticity, and those with suction rapidly weaken by its removal.

The initial state corresponding to Bateman's solution (5.7) is rather more subtle. There, as r approaches zero at some fixed t, the circulation $2\pi r v$ is proportional to $(r^2/t)^{s+1}t^{-1}$and the vorticity to $(r^2/t)^s t^{-2}$. But the limit as t and r both approach zero is non-uniform, depending on the ratio r^2/t. A similar non-uniform limit occurs in the set of solutions (5.12)–(5.13); but, for these, the various Q_n are proportional to t^{s-n} at $r = 0$ with finite t. Again, suction or injection at $r = 0$ combines with viscous diffusion in all cases, yielding solutions with vorticity and circulation that approach zero at sufficiently large r for all $t > 0$.

The various Q_n solutions, though more complex, are similarly affected by injection or removal of vorticity at the axis $r = 0$. Their various reversals in azimuthal velocity are present from the outset, for appropriate values of s, and are imposed by notional initial conditions; the radii at which these occur increase as $t^{1/2}$ on account of viscous diffusion. No matter how great the suction on the axis, these locations always move outwards – a counter-intuitive result, perhaps, but inward motion would be incompatible with the structure of the similarity solutions.

Rather different are the algebraic solutions (5.16)–(5.18) that derive from the C_2 part of (5.11) when s is a positive integer, and also the corresponding algebraic

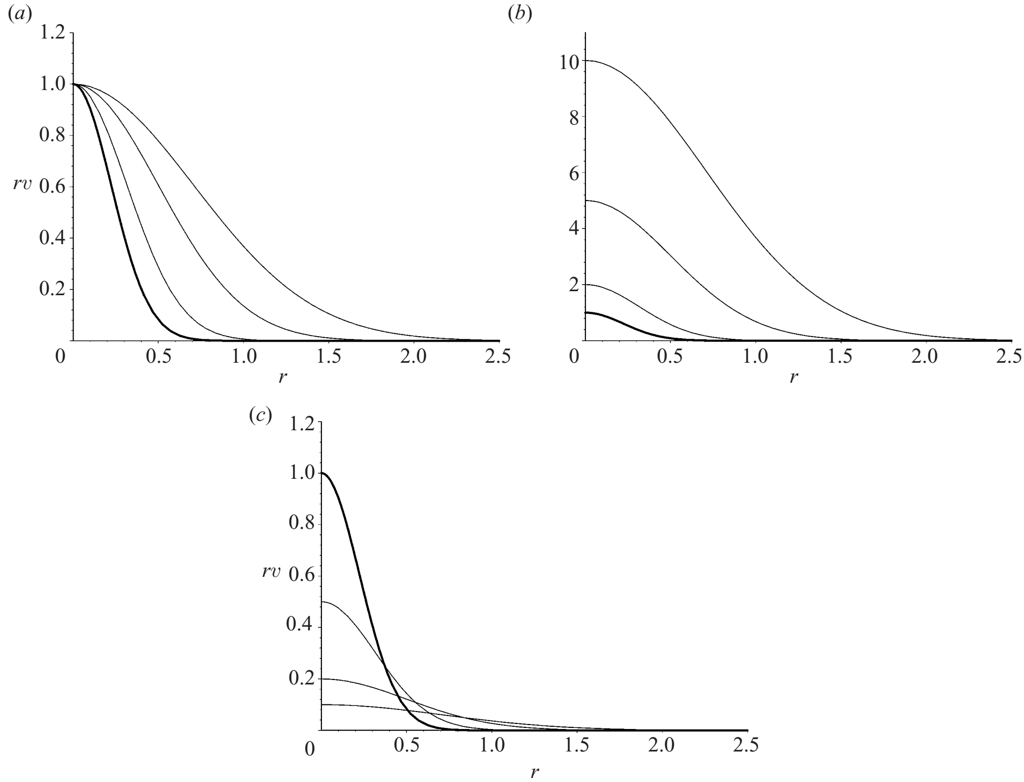

FIGURE 2. The simple solution (5.12) for rv versus r at $\tau = 0.1$, 0.2, 0.5, 1.0 for (*a*) $s = 0$, (*b*) $s = 1$ (blowing) and (*c*) $s = -1$ (suction). To aid comparison, solutions in (*b*) and (*c*) are multiplied by 10 and 0.1 respectively, giving the same function of r for all three at $\tau = 0.1$ (shown by the thicker line).

part of our generalization (5.20) of Bateman's solution. These do not decay to zero exponentially with r^2 as r increases, but may be relevant for flows confined within a cylindrical outer wall of specified radius. The simplest cases to envisage (but not to realize in practice!) have a moving outer boundary at some $r = ct^{1/2}$ corresponding to a constant value of η.

Many such flows between moving cylindrical boundaries may easily be constructed using the above solutions. Consider, for instance, the solutions (5.13) that may exhibit reversals of the flow direction in v, some of which are shown in figure 1. The Q_2 solution exhibits two such reversals when s is negative, at say $\eta = \eta_1$ and η_2, where $\eta = r^2/4\nu t$. Now, if one envisages two expanding cylinders at η_1 and η_2, then the Q_2 solution yields a flow between them that satisfies the no-slip condition on v at both boundaries (without need for their rotation), and with suction or injection of u at both.

It is of some physical interest that (4.2) for rv with constant s may be transformed into (4.3) or (4.5) for diffusion of heat in a longitudinal bar having spatially varying conductivity: this enables solutions such as (4.4) and (4.6) to be found from their thermal equivalents. Also, a referee has pointed out some similarities between our vorticity equation and the Fokker–Planck equation of Brownian motion (see e.g. Uhlenbeck & Ornstein, 1930) that may warrant further investigation.

Some particularly simple separable solutions for rv, when both s and γ are non-zero constants, are given in (3.5). The second and fourth of these exhibit exponential decay as r^2 approaches infinity provided γ is negative. The second has exponential decay with time at each fixed r, while the fourth has constant circulation on circuits of radius r that are expanding or contracting according to $r^2 = 4\ s\nu t\ +$ constant. When s and γ are opposite in sign, the radial velocity changes sign at the radius $r = (-s/\gamma)^{1/2}$; accordingly, solutions with constant s and γ have a 'two-cell' radial structure reminiscent of, but simpler than, that studied by Sullivan (1959).

The most valuable tool for finding solutions with non-zero s and γ is undoubtedly the Rott–Lundgren–Kambe transformation (2.5) (with lower limits of integration changed as necessary), or the generalized similarity method sketched at the end of the previous section. In principle, this gives solutions for *any* $\gamma(t)$ and constant s whenever the corresponding solution with the same s and with $\gamma = 0$ is known. Some explicit solutions are given in (5.23) with $\gamma(t) = \Gamma/t$ and in (5.26) with γ constant. A class of solutions for any $\gamma(t)$ is given in (5.27)–(5.29) and we have described how more may be constructed. Also, derivatives and integrals with respect to time of known solutions of the governing equations (2.3) and (2.4) are themselves solutions; this is a very useful device for deriving further classes.

The various solutions described above demonstrate the interplay of competing physical mechanisms: injection of vorticity with 'blowing' at the inner boundary, or removal of existing vorticity with suction; advection inwards or outwards by the radial velocity component due to the rate-of-strain γ; vortex stretching or contraction due to the corresponding axial rate of strain; and viscous diffusion throughout the flow domain. Each solution can be understood physically in these terms; but it must be remembered that, in assuming particular simple forms, whether separable or self-similar, not all physical scenarios are captured. Though many of these solutions may lack direct practical application, one must be grateful for finding *any* simple exact solutions of the Navier–Stokes equations, as not a huge variety is known.

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